A SOLUTION OF A PROBLEM BY M. HENRIKSEN AND R. G. WOODS

Z. Piotrowski

Abstract. Henriksen and Woods (Problem (C) of [HW] p. 203) asked whether there are Tychonoff spaces X and Y with $X \times Y$ being Baire such that:

(a) Every separately continuous function $f : X \times Y \to \mathbb{R}$ has a dense (in fact: G_δ) set $C(f)$ of points of continuity, and
(b) There exists a separately continuous function $g : X \times Y \to \mathbb{R}$ for which $C(g)$ fails to contain either $A \times Y$ or $X \times B$ for any dense G_δ set $A \subset X$ or dense G_δ set $B \subset Y$.

We will answer this question by showing the spaces X and Y can even be complete metric and condition (b) can be strengthened to the following: There exists a separately continuous function $g : X \times Y \to \mathbb{R}$ so that if $C(g)$ contains either $A \times Y$ or $X \times B$, then both A and B are empty.

1. "Big quadrant"

Let $X = Y = \bigoplus_{\alpha \in [0,1]} [0,1]_\alpha$ be the topological sum of spaces $[0,1]_\alpha$, $\alpha \in [0,1]$ metrized with the metric d, defined as follows:

$$d(x,y) = \begin{cases} |x-y|, & \text{if both } x \text{ and } y \text{ belongs to the same } [0,1]_\alpha, 0 \le \alpha \le 1 \\ 1, & \text{otherwise} \end{cases}$$

(X,d) is obviously a complete metric space. Moreover, we can think of X as an ordered set: within each $[0,1]_\alpha$ we have the usual ordering and if $x \in [0,1]_\alpha$, $y \in [0,1]_\beta$ with $\alpha < \beta$ then $x < y$. Similarly for Y.

Now consider $X \times Y = (\bigoplus_{\alpha \in [0,1]} [0,1]_\alpha) \times (\bigoplus_{\alpha \in [0,1]} [0,1]_\alpha)$; think of it as a matrix consisting of $c \times c$ many squares $S_{r,s} = [0,1]_r \times [0,1]_s$, $0 \le r \le 1$, and $0 \le s \le 1$, having c many "rows" and c many "columns". Each square $S_{r,s}$ has its own local coordinate system.

2. Condition (a)

By the Kuratowski-Montgomery theorem [P], Theorem 3.3 p. 299, any separately continuous function f is class 1 of Baire as a real-valued separately continuous function defined on a product of two metric spaces. As such, f is pointwise discontinuous; that is, the set $D(f)$ of discontinuity points is of first category. Thus $C(f)$ is residual. As a product of two complete metric spaces, the product $X \times Y$ is Baire, so $C(f)$, in fact, *is* a dense G_δ subset of $X \times Y$, since the range of f, the reals, is a metric space. Therefore condition (a) mentioned in Abstract is met.

Typeset by $\mathcal{AMS}$-TEX

3. Condition (b)

We shall now prove that there is a separately continuous function $g : X \times Y \to \mathbb{R}$ for which $C(g)$ contains neither $A \times Y$ nor $X \times B$ for any dense G_δ set $A \subset X$ or dense G_δ set $B \subset Y$.

In what follows we shall construct a set $D \subset X \times Y$ of the form $D = \{D_{r,s} : (r, s) \in [0, 1] \times [0, 1]\}$ where for a fixed pair (r, s), $D_{r,s}$ is a point from the square $S_{r,s}$ lying on its main diagonal, i.e., in the local coordinate system of $S_{q,r}$, $D_{r,s} = (d_{r,s}, d_{r,s})$. We will define the numbers $d_{r,s}$ in such a way that the following holds:

a) $\forall r, s : \mathrm{card}(D \cap S_{r,s}) = 1$

b) $\mathrm{pr}_X\, D$ is dense open in X and $\mathrm{pr}_Y\, D$ is dense open in Y

Let us consider first the uncountable family of squares $S_{r,0}$ lying in the first "row". In the first square $S_{0,0}$ of this family pick $D_{0,0} = (d_{0,0}, d_{0,0}) = (0, 0)$, in the local coordinate system of $S_{0,0}$. Thus $D_{0,0}$ is the lower, left corner. As we increase r, keeping $s = 0$, the point $D_{r,0}$ is gradually moving upwards along the main diagonal of each square until it hits (1,1). More precisely, we put $d_{r,0} = r$, $r \in [0, 1]$. Now fix $s_0 \in (0, 1]$ and consider the corresponding row of the squares. Put

$$d_{r,s_0} = \begin{cases} r + s_0, & \text{if } r + s_0 \le 1 \\ (r + s_0) - 1, & \text{if } r + s_0 > 1. \end{cases}$$

Thus D_{0,s_0} is a point from the diagonal of S_{0,s_0} different from $(0, 0)$ (in local coordinates). As we increase r, keeping s_0 fixed, the point D_{r,s_0} is gradually moving upwards along the main diagonal of each square until it hits (1,1). Then it falls down-left and starts growing from right outside of $(0, 0)$ until it reaches its starting position.

Now we are going to define $g : X \times Y \to \mathbb{R}$ by defining its restriction $g_{r,s}$ to each square $S_{r,s}$ as follows (we use local coordinates):

$$g_{r,s}(x, y) = \begin{cases} \frac{2(x-d_{r,s})(y-d_{r,s})}{(x-d_{r,s})^2+(y-d_{r,s})^2}, & \text{if } (x, y) \ne (d_{r,s}, d_{r,s}) \\ 0, & \text{otherwise} \end{cases}$$

Observe that $g_{r,s}$ is continuous on $S_{r,s}$, except for the point $D_{r,s}$.

It follows from the construction that $C(g)$ contains neither $A \times Y$ nor $X \times B$ for any nonempty set $A \subset X$ or nonempty set $B \subset Y$.

<u>Comment</u>. A somewhat less involved example, one "column" only, was designed by Jack B. Brown (see [P], Example 6.14 p. 313) to answer in *the negative*, questions by A. Alexiewicz, W. Orlicz (see reference [AO] in [P]) and J. P. R. Christensen (see [Cr1] in [P]) whether the assumption that *both* spaces X and Y are complete metric, suffices in Namioka-type theorems. In other words, there are complete metric spaces X and Y and a separately continuous function $f : X \times Y \to \mathbb{R}$ such that there is *no* G_δ set $A \subset X$ such that $A \times Y \subset C(f)$; in fact the largest such a set is empty.

References

[HW] M. Henriksen, R. G. Woods, *Separate versus joint continuity: A tale of four topologies*, Top. Appl. **97** (1999), 175–205.

[P] Z. Piotrowski, *Separate and joint continuity*, Real Analysis Exchange **11** (1985-86), 293–322.